JOURNAL OF APPROXIMATION THEORY 64, 119–121 (1991)

Note

A Short Proof That Every Weak Tchebycheff System May Be Transformed into a Weak Markov System

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Communicated by E. W. Cheney

Received November 17, 1988; revised November 5, 1989

Let $n \geqslant 0$ be an integer and let A be a set of real numbers containing at least $n+2$ elements. A linearly independent sequence $\{y_0, ..., y_n\}$ of real valued functions defined on A is called a *weak Tchebycheff system* (WT-system), provided that for any $t_0 < t_1 < \cdots < t_n$ in A, $\det[y_i(t_j); i, j = 0, ..., n] \geqslant 0$ (cf., e.g., Karlin and Studden [1]). If $\{y_0, ..., y_i\}$ is a WT-system for $i = 0, ..., n$, we say that $\{y_0, ..., y_n\}$ is a *weak Markov system*, or a *complete WT-system*. In [3], Stockenberg proved the following

THEOREM. *Let $Y := \{y_0, ..., y_n\}$ be a WT-system on A. Then there exists a weak Markov system $\{z_0, ..., z_n\}$ spanning the same space as Y.*

A shorter proof of this theorem is due to Schumaker (cf. [2, Theorem 2.41]). However, both proofs are based on a theorem of Zielke that characterizes WT-systems in terms of strong alternations (see, e.g., [5, pp. 12–14]). The purpose of the present paper is to present a short and simple proof of Stockenberg's theorem that only uses the above definition of a WT-system. The method of proof is adapted from [4]. We shall use the following notation:

$$\begin{vmatrix} y_0, ..., y_n \\ t_0, ..., t_n \end{vmatrix} := \det[y_i(t_j); i, j = 0, ..., n].$$

Proof of the Theorem. We may assume $n > 0$. Since Y is linearly independent, there are points $q_0 < q_1 < \cdots < q_n$ in A such that $D := \left|\begin{smallmatrix} y_0, ..., y_n \\ q_0, ..., q_n \end{smallmatrix}\right| > 0$.

0021-9045/91 $3.00

Let $v_i(t) := D^{-1} \left|{}^{y_0, ..., y_n}_{q_0, ..., q_{i-1}, t, q_{i+1}, ..., q_n}\right|$, $i = 0, ..., n$. If

$$\begin{bmatrix} v_0 \\ \vdots \\ v_n \end{bmatrix} = M \begin{bmatrix} y_0 \\ \vdots \\ y_n \end{bmatrix}$$

then $\left|{}^{v_0, ..., v_n}_{t_0, ..., t_n}\right| = \det(M) \left|{}^{y_0, ..., y_n}_{t_0, ..., t_n}\right|$. Thus, since $\left|{}^{v_0, ..., v_n}_{q_0, ..., q_n}\right| = 1$, we deduce that $\det(M) = 1/D > 0$, and therefore that $\{v_0, ..., v_n\}$ is a WT-system on A that spans the same space as Y. Let $u_n := (-1)^n v_0 + (-1)^{n-1} v_1 + \cdots + v_n$, and let B denote the subset of points of $[q_n, \infty) \cap A$ for wich $u_n(t) > 0$. Note that $u_n(q_n) = v_n(q_n) = 1$, and therefore that B is not empty. Moreover, $\{v_0, ..., v_{n-1}, u_n\}$ is a WT-system that spans the same space as Y. Let $b := \sup(B)$, and let $\{b_m\}$ be an increasing sequence of points of B that converges to b. In particular, if $b \in B$ we set $b_m = b$. Note that $u_n(b_m) > 0$. Setting $c_i(m) := v_i(b_m)/u_n(b_m)$ and $u_i(t, m) := v_i(t) - c_i(m)\, u_n(t)$, $i = 0, ..., n-1$, we deduce that

$$u_i(b_m, m) = 0, \qquad i = 0, ..., n-1. \tag{1}$$

It follows from their definition that the functions $(-1)^{n-i} v_i$ are non-negative on $[q_n, \infty) \cap A$, and therefore that $(-1)^{n-i} c_i \geqslant 0$. Since $\sum_{i=0}^{n} (-1)^{n-i} c_i(m) = 1$, this implies that the coefficients $c_i(m)$ are bounded between -1 and 1. Thus there is a sequence $\{m_k\}$ and numbers $c_0, ..., c_{n-1}$ such that $\lim_{k \to \infty} c_i(m_k) = c_i$, $i = 0, ..., n-1$. Let

$$u_i := v_i - c_i u_n, \qquad i = 0, ..., n-1. \tag{2}$$

Since $\{u_0, ..., u_n\}$ spans the same space as Y on A, we conclude that the functions $\{u_0, ..., u_{n-1}\}$ are linearly independent there.

Let $t_0 < t_1 < \cdots < t_{n-1}$ be points in $(-\infty, b) \cap A$. For m sufficiently large we have $b_m > t_{n-1}$, and from (1) we deduce that

$$\left|\begin{matrix} u_0(\cdot, m), ..., u_{n-1}(\cdot, m), u_n \\ t_0, ..., t_{n-1}, b_m \end{matrix}\right| = u_n(b_m) \left|\begin{matrix} u_0(\cdot, m), ..., u_{n-1}(\cdot, m) \\ t_0, ..., t_{n-1} \end{matrix}\right|,$$

and therefore

$$\left|\begin{matrix} u_0(\cdot, m), ..., u_{n-1}(\cdot, m) \\ t_0, ..., t_{n-1} \end{matrix}\right| \geqslant 0.$$

Thus, since $u_i(t) = \lim_{k \to \infty} u_i(t, m_k)$, $i = 0, ..., n-1$, it is clear that

$$\left|\begin{matrix} u_0, ..., u_{n-1} \\ t_0, ..., t_{n-1} \end{matrix}\right| \geqslant 0. \tag{3}$$

Moreover, from the definition of $\{b_m\}$ and (1) we conclude that if $b \in B$ then

$$u_i(b) = 0, \qquad i = 0, ..., n. \tag{4}$$

On the other hand, if $b \in A - B$ then $u_n(b) = 0$ and therefore, since $(-1)^{n-i} v_i(b) \geqslant 0$, we conclude that $v_i(b) = 0$, $i = 0, ..., n-1$, and applying (2) we deduce that (4) is valid in this case as well. Thus, we have shown that (3) is valid for any choice of points $t_0 < \cdots < t_{n-1}$ in $(-\infty, b] \cap A$. If $(b, \infty) \cap A = \varnothing$ we are done. Otherwise note that, since u_n vanishes to the right of b, all the functions v_i must vanish there. Hence, (2) implies that $u_i \equiv 0$ on $(b, \infty) \cap A$ for $i = 0, ..., n-1$. We have therefore shown that $\{u_0, ..., u_{n-1}\}$ is a WT-system on A. Repeating the above procedure for $\{u_0, ..., u_{n-1}\}$ and so on, the conclusion follows. Q.E.D.

References

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Printed in Belgium